Iterates of mappings which are almost continuous and open

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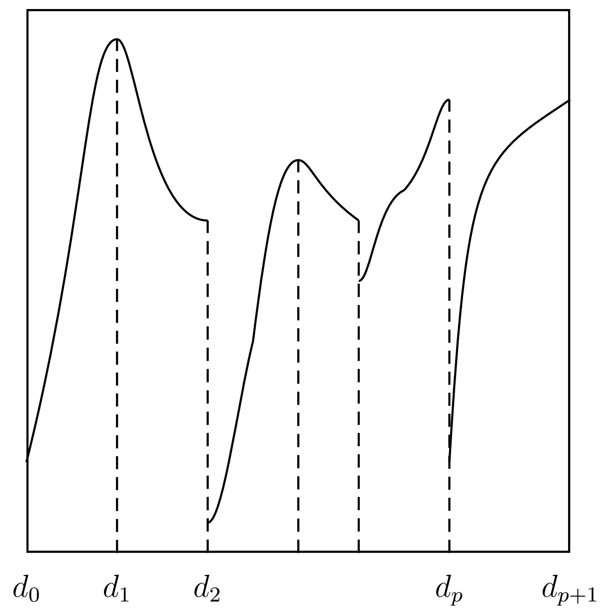
This note presents an approach to studying the iterates of a mapping whose restriction to the complement of a finite set is continuous and open. The main examples to which the approach can be applied are piecewise monotone mappings defined on an interval or a finite graph.

If X is a set and $g : X \to X$ a mapping then for each $n \ge 0$ the *n-th iterate of g* will be denoted by g^n, i.e., $g^n : X \to X$, $n \ge 0$, are the mappings defined inductively by $g^0 = \mathrm{id}_X$ and $g^n = g \circ g^{n-1}$ for each $n \ge 1$. Such a mapping g will be thought of as describing a discrete dynamical system, and in this interpretation the sequence $\{g^n(x)\}_{n\ge 0}$ is the *orbit of the point $x \in X$ under g*.

We work here with a class of mappings which are defined as follows: Let X be a topological space and $g : X \to X$ be a mapping. If O is an open subset of X then g will be called *regular on O* if the restriction of g to O, considered as a mapping from O to X, is both continuous and open (O being endowed with the subspace topology). Hence, since O is open, the requirement is that $g^{-1}(U) \cap O$ and $g(U \cap O)$ must both be open subsets of X for each open $U \subset X$. If g is regular on each set in a family of open sets then it is clearly also regular on their union. There is thus a largest open set Γ_g on which g is regular, and we say that g is *almost regular* if this set Γ_g is dense, i.e., if its closure is the whole of X.

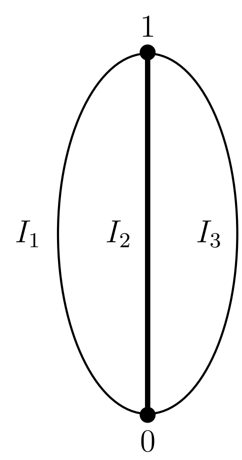
Piecewise monotone mappings of an interval are almost regular: Let $a, b \in \mathbb{R}$ with $a < b$ and put $I = [a, b]$. A mapping $h : I \to I$ is said to be *piecewise monotone* if there exists $p \ge 0$ and $a = d_0 < d_1 < \cdots < d_p < d_{p+1} = b$ such that h is continuous and strictly monotone on each of the open intervals (d_k, d_{k+1}), $k = 0, \ldots, p$. Then h is clearly regular on each of these intervals and hence it is regular on their union. But the complement of the union is the finite set $S_h = \{d_0, \ldots, d_{p+1}\}$ and thus h is almost regular with $\Gamma_h \supset I \setminus S_h$. Note that h is not assumed to be continuous at the points $d_0, d_1, \ldots, d_{p+1}$, although the continuous case is much simpler to deal with.

There is a vast literature on such mappings. The topic in which we are interested (the asymptotic behaviour of 'typical' orbits) is dealt with, for example, in Collet



and Eckmann [3], Guckenheimer [4], Hofbauer [5] and [6], Preston [10] and [11] and Willms [13].

A generalisation of interval mappings are mappings defined on a graph with finitely many edges. Here is an example: For $i = 1, 2, 3$ let $I_i = \{i\} \times [0, 1]$ and let G be the quotient space $I_1 \cup I_2 \cup I_3/\sim$, where there are only two non-trivial equivalence classes given by $(1, 0) \sim (2, 0) \sim (3, 0)$ and $(1, 1) \sim (2, 1) \sim (3, 1)$.



Define a mapping $h : G \to G$ by

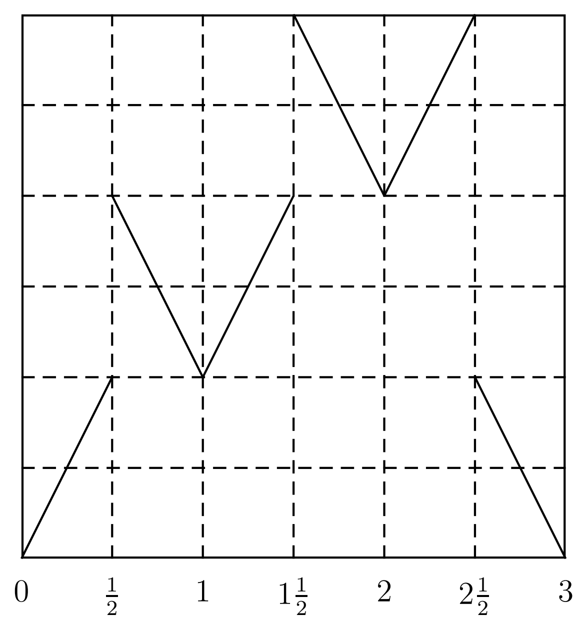
$$h((a, x) = \begin{cases} (a, 2x) & \text{if } x \in [0, \frac{1}{2}] \, , \\ (\sigma(a), 2 - 2x) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

for $a = 1, 2, 3$, where $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$. Then h is almost regular with $\Gamma_h = G \setminus \{(1, \frac{1}{2}), (2, \frac{1}{2}), (3, \frac{1}{2})\}$. Moreover, h is continuous. This mapping h

is more-or-less equivalent to the piecewise monotone mapping $h' : I \to I$, where $I = [0,3]$ and

$$h'(x) = \begin{cases} 2x - a & \text{if } x \in (a, a+\frac{1}{2}),\ a = 0,\ 1,\ 2\ , \\ 3(a+1) - 2x & \text{if } x \in (a+\frac{1}{2}, a+1),\ a = 0,\ 1\ , \\ 6 - 2x & \text{if } x \in (2\frac{1}{2}, 3)\ , \end{cases}$$

and where h' is assigned arbitrary values at the arguments $0,\ \frac{1}{2},\ 1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{2},\ 3$. For $k = 1,\ 2,\ 3$ the interval $[k-1, k] \subset I$ corresponds to the edge I_k in G.



However, when working with the mapping h' it is not easy to make use of the additional information of h being continuous.

There is a now also a large literature on piecewise monotone mappings defined on finite graphs, see, for example, Alseada, Llibre and Misiurewicz [1], Barge and Diamond [2] or Llibre and Misiurewicz [8].

Each non-constant rational mapping $g : \Sigma \to \Sigma$ of the Riemann sphere Σ into (and thus onto) itself is regular, i.e., it is almost regular with $\Gamma_g = \Sigma$. However, the framework to be developed below does not really have much to say about this case.

Given an almost regular mapping $g : X \to X$, we are interested in describing the asymptotic behaviour of the orbit of a 'typical' point x under g. In what follows X will always be a complete metric space, and 'typical' will be taken in the sense associated with the Baire category theorem. Let us thus review the concepts involved here. (For more information Oxtoby [9] is to be recommended.) The interior of a subset A of X will be denoted by $\mathrm{int}(A)$, its closure by $\overline{A}$ and its

boundary (i.e., the subset $\overline{A} \setminus \mathrm{int}(A)$) by ∂A. Moreover, A is *dense* if $\overline{A} = X$ and *nowhere dense* if $\mathrm{int}(\overline{A}) = \varnothing$. In particular the boundary ∂A of an open or a closed set A is nowhere dense. The set A is said to be *meagre* if it can written as a countable union of nowhere dense sets. A set whose complement is meagre is called a *residual set*; this is the case if and only if it contains a dense G_δ-set, where a G_δ-set is one which can be written as a countable intersection of open subsets of X. The Baire category theory (in its version for complete metric spaces) states that a countable intersection of dense open subsets of X is itself dense, which implies that every residual subset of X is dense. Residual sets are clearly closed under taking countable intersections; they are also closed under taking finite unions.

A statement is considered to hold for 'typical' points in X if it holds on a residual set. For a given almost regular mapping $g : X \to X$ we thus want to make statements about the asymptotic behaviour of the orbit $\{g^n(x)\}_{n\ge 0}$ which hold for all points x lying in some residual subset of X.

In these notes we present an approach for dealing with the iterates of an almost regular mapping g. Most of the results hold without any further assumptions. However, in one of the main results (Theorem 3) $X \setminus \Gamma_g$ is required to be a finite set (which is the case for piecewise monotone mappings).

We also assume that X is separable, and so there is a countable base for the topology, and that X is perfect (meaning that there are no isolated points). This ensures that each finite subset of X is nowhere dense.

Lemma 1 *If $B \subset X$ is nowhere dense then so is $g^{-1}(B)$.*

Proof Put $U = \Gamma_g$ and first consider a closed subset E of X. Then $g^{-1}(E) \cap U$ is closed in the subspace topology on U and so there exists a closed subset F of X such that $g^{-1}(E) \cap U = F \cap U$; in particular $g^{-1}(E) \subset F \cup (X \setminus U)$ and thus $\overline{g^{-1}(E)} \subset F \cup (X \setminus U)$. Suppose now there exists a non-empty open set O with $O \subset \overline{g^{-1}(E)}$. Then $O \cap U \subset (F \cup (X \setminus U)) \cap U = F \cap U = g^{-1}(E) \cap U \subset g^{-1}(E)$ and therefore $g(O \cap U) \subset E$. But $g(O \cap U)$ is open and $O \cap U \neq \varnothing$ (since $X \setminus U$ is nowhere dense), and hence $\mathrm{int}(E) \neq \varnothing$. This shows that if $\mathrm{int}(E) = \varnothing$ then $\mathrm{int}(\,\overline{g^{-1}(E)}\,) = \varnothing$. It follows that if B is nowhere dense, i.e., $\mathrm{int}(\overline{B}) = \varnothing$ then $\mathrm{int}(\,\overline{g^{-1}(\overline{B})}\,) = \varnothing$, which implies that $\mathrm{int}(\,\overline{g^{-1}(B)}\,) = \varnothing$, i.e., that $g^{-1}(B)$ is nowhere dense. □

For each $A \subset X$ put $\Lambda_g(A) = \{x \in X : g^n(x) \notin A \text{ for all } n \ge 0\}$; then $\Lambda_g(A)$ is g-invariant, meaning that $g(\Lambda_g(A)) \subset \Lambda_g(A)$, and thus the orbit $\{g^n(x)\}_{n\ge 0}$ of each point $x \in \Lambda_g(A)$ remains in $\Lambda_g(A)$. Moreover, if $S \subset X$ is nowhere dense then by Lemma 1 the set $X \setminus \Lambda_g(S) = \bigcup_{n\ge 0} g^{-n}(S)$ is meagre, and so $\Lambda_g(S)$ is a residual set.) Therefore, since we are only interested in the asymptotic behaviour

of the orbits of 'typical' points in X, we can choose to ignore what happens with the orbits of the points in $X \setminus \Lambda_g(S)$. In particular, this means that it doesn't matter if we change the mapping g on a nowhere dense set S, since this set is avoided by the orbits of all the points in the residual set $\Lambda_g(S)$.

We choose to modify the mapping g as follows: Fix a closed nowhere dense set S with $X \setminus \Gamma_g \subset S$, let $\bullet$ be some element not in X, put $X_\bullet = X \cup \{\bullet\}$ and define a mapping $f : X_\bullet \to X_\bullet$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in X \setminus S \,, \\ \bullet & \text{if } x \in S \cup \{\bullet\} \,. \end{cases}$$

Thus $\Lambda_f(S) = \{x \in X : f^n(x) \notin S \text{ for all } n \geq 0\} = \Lambda_g(S)$, by Lemma 1 $\Lambda_f(S)$ is a residual set, and for each $x \in \Lambda_f(S)$ the orbit of x under f is the same as its orbit under g. In the applications S will always be taken to be $X \setminus \Gamma_g$, although taking S somewhat larger may sometimes give additional information. It turns out to be much more convenient to work with f than with the mapping g, and so this is what we will do.

The set $X_\bullet$ will not be endowed with a topology, and statements of a topological nature always refer to the topology on X, and thus must be statements about subsets of X. For example, the statement that a set O is open means that O is an open subset of X (and in particular includes the assertion that $O \subset X$). Since $X \setminus S$ is open the statement that O is an open subset of $X \setminus S$ means that O is a subset of $X \setminus S$ which is an open subset of X.

From the definition of f it follows that $f(X \setminus S) \subset X$ and that the restriction of f to $X \setminus S$ is equal to the restriction of g to $X \setminus S$. Thus the restriction of f to $X \setminus S$, considered as a mapping from $X \setminus S$ to X, is both continuous and open (where $X \setminus S$ is again given the subspace topology). More explicitly, this means:

(A_1) $f^{-1}(O)$ is an open subset of $X \setminus S$ for each open set O.

(A_2) $f(O \setminus S)$ is open for each open set O.

Condition (A_1) holds because $f^{-1}(A) = g^{-1}(A) \setminus S$ for all $A \subset X$, and (A_2) holds because $f(A \setminus S) = g(A \setminus S)$ for all $A \subset X$.

Since $f^{-1}(X) \subset X \setminus S$ it follows by induction that $f^{-n}(X) \subset X \setminus S$ for all $n \geq 1$. It then also follows from (A_1) by induction that if O is open then $f^{-n}(O)$ is an open subset of $X \setminus S$ for each $n \geq 1$.

For each $A \subset X$ we denote the set $f(A \setminus S)$ by $f_{\setminus S}(A)$ (although some care is needed here, since there is no mapping $f_{\setminus S}$ involved). In particular (A_2) implies that $f_{\setminus S}(O)$ is an open subset of $X \setminus S$ for each open set O.

A subset A of X will be called $f_{\setminus S}$*-invariant* if $f_{\setminus S}(A) \subset A$. Being more explicit, this means that $f(A \setminus S) \subset A$ must hold, and note that $f(A \setminus S) = X \cap f(A)$.

The subset A is $f_{\setminus S}$-invariant if and only if $A \cup \{\bullet\}$ is f-invariant, i.e., if and only if $f(A \cup \{\bullet\}) \subset A \cup \{\bullet\}$. This holds because if $f(A \cup \{\bullet\}) \subset A \cup \{\bullet\}$ then

$$f(A \setminus S) = X \cap f(A) \subset X \cap f(A \cup \{\bullet\}) \subset X \cap (A \cup \{\bullet\}) = A \ ,$$

and if $f(A \setminus S) \subset A$ then $f(A \cup \{\bullet\}) = f(A \setminus S) \cup \{\bullet\} \subset A \cup \{\bullet\}$.

The set $\Lambda_f(S) = \{x \in X : f^n(x) \notin S \text{ for all } n \geq 0\}$ is $f_{\setminus S}$-invariant and for each $A \subset X$ the set $A \cap \Lambda_f(S)$ is $f_{\setminus S}$-invariant if and only if it is g-invariant.

The sets $\varnothing$ and X are clearly $f_{\setminus S}$-invariant and arbitrary unions and intersections of $f_{\setminus S}$-invariant sets are again $f_{\setminus S}$-invariant.

Lemma 2 *If A is $f_{\setminus S}$-invariant then $X \cap f^n(A) \subset A$ for all $n \geq 1$. Moreover, if O is open then so is $X \cap f^n(O)$.*

Proof Since $X \cap f(B) = X \cap f(X \cap B)$ for all $B \subset X$ we have

$$X \cap f^{n+1}(A) = X \cap f(f^n(A)) = X \cap f(X \cap f^n(A)) = f_{\setminus S}(X \cap f^n(A))$$

for all $A \subset X$ and all $n \geq 0$. If A is invariant then $X \cap f(A) = f_{\setminus S}(A) \subset A$ and so it follows from the above and by induction on n that $X \cap f^n(A) \subset A$ for all $n \geq 1$. The final statement also follows by induction since $f_{\setminus S}(O')$ is open for each open set O'. □

Lemma 3 *For each subset A of X the complement $X \setminus A$ is $f_{\setminus S}$-invariant if and only if $f^{-1}(A) \subset A$.*

Proof The set $X \setminus A$ is $f_{\setminus S}$-invariant if and only if $f((X \setminus A) \setminus S) \subset X \setminus A$ and, since $(X \setminus A) \setminus S = X \setminus (A \cup S)$, this holds if and only if $f(x) \in X \setminus A$ whenever $x \in X \setminus (A \cup S)$, and thus if and only if $x \in A \cup S$ whenever $f(x) \in A$. Hence $X \setminus A$ is $f_{\setminus S}$-invariant if and only if $f^{-1}(A) \subset A \cup S$. But $f^{-1}(A) \subset A \cup S$ holds if and only if $f^{-1}(A) \subset A$ does, since $f^{-1}(A) \subset X \setminus S$. □

If $X \setminus A$ is $f_{\setminus S}$-invariant then it follows from Lemma 3 and by induction on n that $f^{-n}(A) \subset A$ for all $n \geq 1$, since $f^{-(n+1)}(A) = f^{-1}(f^{-n}(A))$ for all $n \geq 0$.

The following lemma gives the two properties of $f_{\setminus S}$-invariant sets which play a fundamental role in the analysis of the iterates of f.

Lemma 4 *If a subset A of X is $f_{\setminus S}$-invariant then so are its interior* $\mathrm{int}(A)$ *and its closure $\overline{A}$.*

Proof Let A be $f_{\setminus S}$-invariant; then $f_{\setminus S}(\mathrm{int}(A)) \subset f_{\setminus S}(A) \subset A$ and by (A_2) the set $f_{\setminus S}(\mathrm{int}(A))$ is open. Hence $f_{\setminus S}(\mathrm{int}(A)) \subset \mathrm{int}(A)$, i.e., $\mathrm{int}(A)$ is $f_{\setminus S}$-invariant. Now consider $x \in \overline{A} \setminus S$, and let O be any open set containing $f(x)$. Then x lies in the set $f^{-1}(O)$, which by (A_1) is open. Hence $f^{-1}(O) \cap A \neq \varnothing$ (since $x \in \overline{A}$), and so there exists $y \in f^{-1}(O) \cap A$. In particular $y \in A \setminus S$ (since $f(y) \in O \subset X$), and it follows that $f(y) \in A$, since A is $f_{\setminus S}$-invariant, which implies that $f(y) \in O \cap A$. Therefore A intersects each open set containing $f(x)$, and thus $f(x) \in \overline{A}$. This shows that $f_{\setminus S}(\overline{A}) = f(\overline{A} \setminus S) \subset \overline{A}$, and hence that $\overline{A}$ is $f_{\setminus S}$-invariant. □

A subset $A \subset X$ will be called *fully* $f_{\setminus S}$*-invariant* if both A and its complement $X \setminus A$ are $f_{\setminus S}$-invariant. The sets $\varnothing$ and X are fully $f_{\setminus S}$-invariant and arbitrary unions and intersections of fully $f_{\setminus S}$-invariant sets are again fully $f_{\setminus S}$-invariant. Moreover, by definition they are closed under taking complements. If A is fully $f_{\setminus S}$-invariant then Lemma 4 shows that the interior $\mathrm{int}(A)$ and the closure $\overline{A}$ of A are also fully $f_{\setminus S}$-invariant, since $X \setminus \mathrm{int}(A) = \overline{X \setminus A}$ and $X \setminus \overline{A} = \mathrm{int}(X \setminus A)$.

The set of all fully $f_{\setminus S}$-invariant open subsets of X will be denoted by $\mathcal{D}_f$. Our aim is to find a decomposition of X into disjoint elements $\{D_\alpha\}$ of $\mathcal{D}_f$ such that the union $\bigcup_\alpha D_\alpha$ is dense in X and such that the behaviour of the iterates of f on each of the components D_α allows some kind of reasonable description.

In fact, it is convenient to only consider the regular open sets in $\mathcal{D}_f$: For each subset $B \subset X$ put $B^\diamond = \mathrm{int}(\overline{B})$; a set O is said to be *regular open* if $O = O^\diamond$. (Such sets were introduced by Kuratowksi in [7] and have proved useful in various settings.) In particular B is nowhere dense if and only if $B^\diamond = \varnothing$. If $A \subset B$ then clearly $A^\diamond \subset B^\diamond$. The elementary properties of the $\diamond$-operation which we will make use of are listed in the following lemma:

Lemma 5 *(1)* $(B^\diamond)^\diamond = B^\diamond$ *for all* $B \subset X$.

(2) A set O *is regular open if and only if* $O = B^\diamond$ *for some* B*, which is the case if and only if* $O = \mathrm{int}(F)$ *for some closed set* F.

(3) If O_1, O_2 *are regular open then so is* $O_1 \cap O_2$.

(4) If O *is open (and in particular if* O *is regular open) then* $X \setminus \overline{O}$ *is regular open.*

(5) If O_1, O_2 *are regular open then so is* $O_2 \setminus \overline{O_1}$. *Moreover, if* O_1 *is a proper subset of* O_2 *then* $O_2 \setminus \overline{O_1}$ *is non-empty.*

(6) If $O \subset X$ *is open then* $O \subset O^\diamond$ *and* $O^\diamond \setminus O \subset \partial O$ *(the boundary of* O*), and in particular* $O^\diamond \setminus O$ *is nowhere dense.*

(7) If O_1, O_2 *are open then* $O_1 \cap O_2 \neq \varnothing$ *if and only if* $O_1 \cap O_2^\diamond \neq \varnothing$ *which is the case if and only if* $O_1^\diamond \cap O_2^\diamond \neq \varnothing$.

Proof (1) $B^\diamond = \mathrm{int}(\overline{B}) = \mathrm{int}(\overline{\mathrm{int}(\overline{B})}) \subset \mathrm{int}(\overline{\mathrm{int}(\overline{B})}) = (B^\diamond)^\diamond$, and on the other hand $(B^\diamond)^\diamond = \mathrm{int}(\overline{\mathrm{int}(\overline{B})}) \subset \mathrm{int}(\overline{\overline{B}}) = \mathrm{int}(\overline{B}) = B^\diamond$.

(2) If $O = B^\diamond$ for some B then $O^\diamond = (B^\diamond)^\diamond = B^\diamond = O$ and so O is regular open. Conversely, if O is regular open then $O = B^\diamond$ with $B = O$. The rest follows since $\mathrm{int}(F) = \mathrm{int}(\overline{F}) = F^\diamond$ whenever F is open.

(3) We have $(O_1 \cap O_2)^\diamond \subset O_1^\diamond = O_1$, since $O_1 \cap O_2 \subset O_1$, and in the same way $(O_1 \cap O_2)^\diamond \subset O_2$. Thus $(O_1 \cap O_2)^\diamond \subset O_1 \cap O_2$. But $O \subset O^\diamond$ for each open set O and so also $O_1 \cap O_2 \subset (O_1 \cap O_2)^\diamond$. Hence $O_1 \cap O_2 = (O_1 \cap O_2)^\diamond$, i.e., $O_1 \cap O_2$ is regular open.

(4) This follows from (2), since $X \setminus \overline{O} = \mathrm{int}(X \setminus O)$ and $X \setminus O$ is closed.

(5) By (3) and (4) $O_2 \setminus \overline{O_1} = O_2 \cap (X \setminus \overline{O_1})$ is regular open. Now if $O_2 \subset \overline{O_1}$ then $O_2 \subset \mathrm{int}(\overline{O_1}) = O_1^\diamond = O_1$ and so O_1 is not a proper subset of O_2. Thus if O_1 is a proper subset of O_2 then $O_2 \setminus \overline{O_1}$ is non-empty.

(6) Since $O \subset \overline{O}$ and O is open it follows that $O \subset \mathrm{int}(\overline{O}) = O^\diamond$. Moreover, $O^\diamond \setminus O \subset \overline{O} \setminus O = \partial O$.

(7) If $O_1 \cap O_2 = \varnothing$ then $O_1 \cap \overline{O_2} = \varnothing$ and hence $O_1 \cap O_2^\diamond = O_1 \cap \mathrm{int}(\overline{O_2}) = \varnothing$. In the same way it then follows that $O_1^\diamond \cap O_2^\diamond = \varnothing$. The converse is clear, since $O_1 \subset O_2^\diamond$ and $O_2 \subset O_2^\diamond$. □

We are working here with open sets where more typically closed sets would be used, and the $\diamond$-operation can be thought of taking on the role normally played by the closure operation.

Denote by $\mathcal{D}_f^\diamond$ the set of non-empty regular open elements in $\mathcal{D}_f$. Two important properties of elements $D_1, D_2 \in \mathcal{D}_f^\diamond$ are the following: If $D_1 \cap D_2 \neq \varnothing$ then by Lemma 5 (3) $D_1 \cap D_2 \in \mathcal{D}_f^\diamond$. Moreover, if D_1 is a proper subset of D_2 then by Lemma 5 (5) $D_2 \setminus \overline{D_1}$ is non-empty, and thus an element of $\mathcal{D}_f^\diamond$.

A set $D \in \mathcal{D}_f^\diamond$ will be called *minimal* if the only element of $\mathcal{D}_f^\diamond$ which is a subset of D is D itself. Equivalently, $D \in \mathcal{D}_f^\diamond$ is minimal if $D \subset E$ whenever $E \in \mathcal{D}_f^\diamond$ with $E \cap D \neq \varnothing$ (since then $E \cap D \in \mathcal{D}_f^\diamond$). This implies that if $D_1, D_2 \in \mathcal{D}_f^\diamond$ are minimal with $D_1 \neq D_2$ then $D_1 \cap D_2 = \varnothing$, i.e., minimal elements are either equal or disjoint. In particular, the set of minimal elements in $\mathcal{D}_f^\diamond$ is countable, since we are assuming X is separable.

Denote by M_f the union of all the minimal elements in $\mathcal{D}_f^\diamond$; thus $M_f \in \mathcal{D}_f$. Put $N_f = X \setminus \overline{M_f}$ and so by Lemma 5 (4) N_f is regular open, which means that N_f is either empty or an element of $\mathcal{D}_f^\diamond$. Moreover, M_f and N_f are disjoint and their union $M_f \cup N_f$ is a dense open subset of X, since $X \setminus (M_f \cup N_f) = \overline{M_f} \setminus M_f$ is the boundary of the open set M_f.

For each $x \in X$ let $\mathcal{D}_f^\diamond(x)$ be the set of all elements in $\mathcal{D}_f^\diamond$ which contain x, and put $G_f(x) = \bigcap_{D \in \mathcal{D}_f^\diamond(x)} \overline{D}$, i.e., $G_f(x)$ is the intersection of the closures of the elements

in $\mathcal{D}_f^\diamond(x)$. Hence $G_f(x)$ is a closed fully $f_{\setminus S}$-invariant set containing x (since $X \in \mathcal{D}_f^\diamond(x)$). We write $G_f^\diamond(x)$ instead of $(G_f(x))^\diamond$; thus $G_f^\diamond(x) = \mathrm{int}(G_f(x))$, since $G_f(x)$ is closed, and $G_f^\diamond(x)$ is either empty or an element of $\mathcal{D}_f^\diamond$.

Here is a more explicit description of the set $G_f(x)$: Since X is fully $f_{\setminus S}$-invariant and arbitrary intersections of fully $f_{\setminus S}$-invariant sets are fully $f_{\setminus S}$-invariant there is a least fully $f_{\setminus S}$-invariant set containing a given subset A of X, and this set will be denoted by $\Delta_f(A)$. If O is open then by Lemma 4 $\Delta_f(O)$ is also open, since then $\mathrm{int}(\Delta_f(O))$ is a fully $f_{\setminus S}$-invariant set containing O. If $D \in \mathcal{D}_f^\diamond(x)$ then $B_{1/n}(x) \subset D$ for some $n \geq 1$ (with $B_r(x)$ the open ball of radius r and centre x) and hence $\Delta_f(B_{1/n}(x)) \subset D$; on the other hand $\Delta_f(B_{1/n}(x)) \in \mathcal{D}_f^\diamond(x)$ for each n. It follows that $G_f(x) = \bigcap_{n\geq 1} \overline{\Delta_f(B_{1/n}(x))}$, which represents $G_f(x)$ as the intersection of a decreasing sequence of sets. For each $A \subset X$ it is convenient to write $\Delta_f^\diamond(A)$ instead of $(\Delta_f(A))^\diamond$.

Theorem 1 *(1) If D is a minimal element of $\mathcal{D}_f^\diamond$ then $G_f^\diamond(x) = D$ for each $x \in D$, and in particular $x \in G_f^\diamond(x)$.*

(2) There exists a meagre subset Z of X such that $G_f^\diamond(x) = \varnothing$ (i.e., such that $G_f(x)$ is nowhere dense) for all $x \in N_f \setminus Z$.

(3) $M_f \subset \mathrm{int}(\{x \in X : G_f^\diamond(x) \neq \varnothing\}) \subset M_f^\diamond$.

Proof (1) If $E \in \mathcal{D}_f^\diamond(x)$ then $D \subset E$, since D is minimal; also, $D \in \mathcal{D}_f^\diamond(x)$. Thus $G_f(x) = \overline{D}$ and hence $G_f^\diamond(x) = D^\diamond = D$.

(2) Let $\{O_n\}_{n\geq 1}$ be a sequence of non-empty open sets such that each non-empty open set contains some O_n (which exists since the topology on X has a countable base). For each $n \geq 1$ put $D_n = \Delta_f^\diamond(O_n)$ and let $Z = \bigcup_{n\geq 1}(\overline{D_n} \setminus D_n)$; then Z is a meagre subset of X, since $\overline{D_n} \setminus D_n$, as the boundary of the open set D_n, is nowhere dense for each n. Now let $x \in N_f \setminus Z$ and suppose $G_f^\diamond(x) \neq \varnothing$; then $G_f^\diamond(x) \subset N_f^\diamond = N_f$, since $D \cap N_f \in \mathcal{D}_f^\diamond(x)$ for each $D \in \mathcal{D}_f^\diamond(x)$. Hence $G_f^\diamond(x)$ is not minimal and so there exists $D \in \mathcal{D}_f^\diamond$ with D a proper subset of $G_f^\diamond(x)$. Let $n \geq 1$ be such that $O_n \subset D$; then $\Delta_f(O_n) \subset D$, since $\Delta_f(O_n)$ is the least fully $f_{\setminus S}$-invariant set containing O_n and thus also $D_n = \Delta_f^\diamond(O_n) \subset D^\diamond = D$. By Lemma 5 (5) $G_f^\diamond(x) \setminus \overline{D} \neq \varnothing$, and $G_f^\diamond(x) \setminus \overline{D_n} \supset G_f^\diamond(x) \setminus \overline{D}$, which shows that $G_f^\diamond(x) \setminus \overline{D_n} \neq \varnothing$. In particular $E_n = X \setminus \overline{D_n}$ is non-empty and so an element of $\mathcal{D}_f^\diamond$. However, x is either in D_n or E_n, since $X \setminus (D_n \cup E_n) = \overline{D_n} \setminus D_n \subset Z$, which means that one of D_n or E_n is an element of $\mathcal{D}_f^\diamond(x)$. Suppose $D_n \in \mathcal{D}_f^\diamond(x)$; then $G_f^\diamond(x) \subset \mathrm{int}(\overline{D_n}) = D_n^\diamond = D_n$, and this is not possible, since D_n is a proper subset of $G_f^\diamond(x)$. But if $E_n \in \mathcal{D}_f^\diamond(x)$ then $G_f^\diamond(x) \subset \mathrm{int}(\overline{E_n}) = E_n^\diamond = E_n$, which again is not possible because $D_n \subset G_f^\diamond(x) \setminus E_n$. It therefore follows that $G_f^\diamond(x) = \varnothing$.

(3) Put $U = \mathrm{int}(\{x \in X : G_f^\diamond(x) \neq \varnothing\})$. By (1) $G_f^\diamond(x) \neq \varnothing$ for all $x \in M_f$ and so $M_f \subset U$. Now let $x \in N_f$ and let O be any open set containing x; then

$O \cap N_f \neq \varnothing$. Thus $O \cap (N_f \setminus Z) = (O \cap N_f) \cap (X \setminus Z) \neq \varnothing$, since the residual set $X \setminus Z$ is dense. It follows that $G_f^\diamond(y) = \varnothing$ for some $y \in O$, which implies that $x \notin U$, and shows that $N_f \cap U = \varnothing$. Therefore $U \subset \mathrm{int}(X \setminus N_f) = X \setminus \overline{N_f}$. But $N_f = X \setminus \overline{M_f}$ and so $X \setminus \overline{N_f} = X \setminus \overline{X \setminus \overline{M_f}} = X \setminus (X \setminus M_f^\diamond) = M_f^\diamond$. □

If $x \in \partial D$ for some minimal element D of $\mathcal{D}_f^\diamond$ then $G_f^\diamond(x) \supset D$, which implies in particular that $G_f^\diamond(x) \neq \varnothing$.

For each $x \in X$ the singleton set $\{x\}$ will be denoted just by x, so for example $\Delta_f(x)$ is the least fully $f_{\setminus S}$-invariant set containing x. It is easily checked that

$$\Delta_f(x) = \{y \in X : \text{there exist } m,\, n \geq 0 \text{ with } f^m(y) \in X \text{ and } f^m(y) = f^n(x)\}\ .$$

Note that if A is any fully $f_{\setminus S}$-invariant set with $A \cap \Delta_f(x) \neq \varnothing$ then $x \in A$. (There exists $y \in A$ and $m,\, n \geq 0$ with $f^m(y) \in X$ and $f^m(y) = f^n(x)$ and thus by Lemma 2 $f^n(x) = f^m(y) \in A$; Lemma 3 then shows that $x \in A$.)

Theorem 1 (2) implies in particular that there exists a meagre subset Z of X such that $\Delta_f(x)$ is nowhere dense for all $x \in N_f \setminus Z$ (since $\overline{\Delta_f(x)} \subset G_f(x)$ for all x). In fact we have the following:

Proposition 1 *The set $\Delta_f(x)$ is nowhere dense for all $x \in N_f$.*

Proof Let $x \in N_f$; then, since $\Delta_f(x) \subset N_f$ it follows that $\Delta_f^\diamond(x) \subset N_f^\diamond = N_f$. Suppose $\Delta_f^\diamond(x) \neq \varnothing$, which means $\Delta_f^\diamond(x)$ is an element of $\mathcal{D}_f^\diamond$ and a subset of N_f. Hence $\Delta_f^\diamond(x)$ is not minimal and so there exists $D \in \mathcal{D}_f^\diamond$ with D a proper subset of $\Delta_f^\diamond(x)$, and then by Lemma 5 (5) $D' = \Delta_f^\diamond(x) \setminus \overline{D}$ is also an element of $\mathcal{D}_f^\diamond$. It follows that each of the open sets D and D' intersects $\overline{\Delta_f(x)}$ and hence they each intersect $\Delta_f(x)$. But D and D' are both fully $f_{\setminus S}$-invariant, which implies they both contain x. This is not possible because they are disjoint, which shows that $\Delta_f^\diamond(x) = \varnothing$. □

We next look at what goes on inside a minimal element of $\mathcal{D}_f^\diamond$, and begin by considering which sets should correspond to the minimal elements of $\mathcal{D}_f^\diamond$ in the case of $f_{\setminus S}$-invariant (rather than fully $f_{\setminus S}$-invariant) sets.

Denote by $\mathcal{I}_f$ the set of all $f_{\setminus S}$-invariant open subsets of X and by $\mathcal{I}_f^\diamond$ the set of non-empty regular open sets in $\mathcal{I}_f$. Note that if $U_1,\, U_2 \in \mathcal{I}_f^\diamond$ with $U_1 \cap U_2 \neq \varnothing$ then by Lemma 5 (3) $U_1 \cap U_2$ is again an element of $\mathcal{I}_f^\diamond$. An element $U \in \mathcal{I}_f^\diamond$ is said to be *transitive* if the only $V \in \mathcal{I}_f^\diamond$ with $V \subset U$ is U itself. Thus U is transitive if and only if $U \subset V$ for each $V \in \mathcal{I}_f^\diamond$ with $V \cap U \neq \varnothing$ (since $V \cap U$ is then an element of $\mathcal{I}_f^\diamond$ with $V \cap U \subset U$). This implies in particular that transitive elements of $\mathcal{I}_f^\diamond$ are either equal or disjoint.

The transitive elements of $\mathcal{I}_f^\diamond$ are, of course, just the minimal elements of $\mathcal{I}_f^\diamond$. The denotation 'transitive' is employed to avoid the confusion of having two different

usages for the term 'minimal' and because the property corresponds to the usual one of being topologically transitive (but expressed here for open rather than closed sets); see, for example, Walters [12].

A decreasing sequence $\{U_n\}_{n\geq 1}$ of sets from $\mathcal{I}_f^\diamond$ will be called an *asymptotically transitive cascade*, or for short just a *transitive cascade*, if for each $V \in \mathcal{I}_f^\diamond$ with $V \cap U_1 \neq \varnothing$ there exists $m \geq 1$ such that $U_m \subset V$. In particular, if $U \in \mathcal{I}_f^\diamond$ is transitive then the constant sequence $\{U_n\}_{n\geq 1}$ with $U = U_n$ for all $n \geq 1$ is a transitive cascade, and it will be denoted by $\{U\}$. We will see that each minimal element of $\mathcal{D}_f^\diamond$ contains an essentially unique transitive cascade and that, conversely, each transitive cascade is contained in a unique minimal element of $\mathcal{D}_f^\diamond$.

For each transitive cascade $\gamma = \{U_n\}_{n\geq 1}$ we put $C_\gamma = \bigcap_{n\geq 1} \overline{U_n}$ and call C_γ the *core of* γ. If X is compact then C_γ is non-empty, but in general this need not be the case. The core C_γ is closed and $f_{\setminus S}$-invariant, and hence $C_\gamma^\diamond = \mathrm{int}(C_\gamma)$ is either empty or an element of $\mathcal{I}_f^\diamond$. Moreover, $C_\gamma^\diamond \subset U_n$ for all $n \geq 1$, since $C_\gamma^\diamond = \mathrm{int}(C_\gamma) \subset \mathrm{int}(\overline{U_n}) = U_n^\diamond = U_n$. If U is a transitive element of $\mathcal{I}_f^\diamond$ then of course $C_{\{U\}} = \overline{U}$ and so $C_{\{U\}}^\diamond = U$.

Proposition 2 *If $\gamma = \{U_n\}_{n\geq}$ is a transitive cascade with $C_\gamma^\diamond \neq \varnothing$ then U_m is transitive for some $m \geq 1$ and $C_\gamma^\diamond = U_m$. In particular, for each transitive cascade γ the set $C_\gamma^\diamond$ is either empty or a transitive element of $\mathcal{I}_f^\diamond$.*

Proof We have $C_\gamma^\diamond \in \mathcal{I}_f^\diamond$ and $C_\gamma^\diamond \cap U_1 = C_\gamma^\diamond \neq \varnothing$. Thus $U_m \subset C_\gamma^\diamond$ for some $m \geq 1$, and thus $U_m \subset C_\gamma^\diamond \subset U_m$, i.e., $C_\gamma^\diamond = U_m$. Now if U is any element of $\mathcal{I}_f^\diamond$ with $U \cap U_m \neq \varnothing$ then $U_p \subset U$ for some $p \geq 1$ and hence $U_m = C_\gamma^\diamond \subset U_p \subset U$. Therefore U_m is transitive. $\square$

Let $\gamma = \{U_n\}_{n\geq}$ be a transitive cascade and consider the family $\mathcal{S}_\gamma$ consisting of those elements $U \in \mathcal{I}_f^\diamond$ which contain U_1 and are such that if $V \in \mathcal{I}_f^\diamond$ with $V \cap U \neq \varnothing$ then $U_m \subset V$ for some $m \geq 1$. (Thus U being in $\mathcal{S}_\gamma$ means that $\{U'_n\}_{n\geq}$ is still a transitive cascade, where $U'_1 = U$ and $U'_{n+1} = U_n$ for all $n \geq 1$.) Now $\mathcal{S}_\gamma$ contains U_1 and thus is non-empty, and so by Lemma 5 (7) the set $D_\gamma = W^\diamond$, where W is the union of all the sets in $\mathcal{S}_\gamma$, is again an element of $\mathcal{S}_\gamma$, and hence it is the largest element. We call D_γ the *domain of* γ.

If U is a transitive element of $\mathcal{I}_f^\diamond$ then we write D_U instead of $D_{\{U\}}$ and call D_U the *domain of* U. Thus D_U is the union of all the sets in the family $\mathcal{S}_U$ consisting of those elements $V \in \mathcal{I}_f^\diamond$ which contain U and are such that if $V' \in \mathcal{I}_f^\diamond$ with $V' \cap V \neq \varnothing$ then $U \subset V'$.

Theorem 2 *The domain of a transitive cascade is a minimal element of $\mathcal{D}_f^\diamond$. Conversely, each minimal element of $\mathcal{D}_f^\diamond$ is the domain of some transitive cascade.*

Proof This requires some preparation. For each subset $A \subset X$ put

$$\Delta_f^-(A) = \{x \in X : f^n(x) \in A \text{ for some } n \geq 0\} ,$$

and hence $\Delta_f^-(A) = \bigcup_{n\geq 0} f^{-n}(A)$, since $f^{-n}(A) \subset X$ for each n. In particular, if O is open then so is $\Delta_f^-(O)$.

Lemma 6 *(1) The set $X \setminus \Delta_f^-(A)$ is $f_{\setminus S}$-invariant for each $A \subset X$.*

(2) The set $X \setminus A$ is $f_{\setminus S}$-invariant if and only if $A = \Delta_f^-(A)$.

(3) If A is $f_{\setminus S}$-invariant then so is $\Delta_f^-(A)$ and $\Delta_f(A) = \Delta_f^-(A)$.

(4) If A and B are $f_{\setminus S}$-invariant sets then $\Delta_f(A) \cap \Delta_f(B) \neq \varnothing$ if and only if $A \cap B \neq \varnothing$, which in turn holds if and only if $A \cap \Delta_f(B) \neq \varnothing$.

Proof (1) $f^{-1}(\Delta_f^-(A)) = \{x \in X : f^n(x) \in A \text{ for some } n \geq 1\} \subset \Delta_f^-(A)$ and so by Lemma 2 $X \setminus \Delta_f^-(A)$ is $f_{\setminus S}$-invariant.

(2) If $A = \Delta_f^-(A)$ then by (1) $X \setminus A$ is $f_{\setminus S}$-invariant. Conversely, if $X \setminus A$ is $f_{\setminus S}$-invariant then by Lemma 2 and induction $f^{-n}(A) \subset A$ for all $n \geq 1$ and hence $A \subset \Delta_f^-(A) = \bigcup_{n\geq 0} f^{-n}(A) \subset A$, i.e., $A = \Delta_f^-(A)$.

(3) Let $x \in \Delta_f^-(A) \setminus S$; then $f^n(x) \in A$ for some $n \geq 0$. If $x \in A$ then $x \in A \setminus S$ and so $f(x) \in A$, since A is $f_{\setminus S}$-invariant. On the other hand, if $f^n(x) \in A$ with $n \geq 1$ then $f^m(f(x)) \in A$ with $m = n - 1$. In both cases $f(x) \in \Delta_f^-(A)$ and so $f_{\setminus S}(\Delta_f^-(A)) \subset \Delta_f^-(A)$, i.e., $\Delta_f^-(A)$ is $f_{\setminus S}$-invariant. Together with (1) this shows $\Delta_f^-(A)$ is fully $f_{\setminus S}$-invariant and hence $\Delta_f(A) \subset \Delta_f^-(A)$, since clearly $A \subset \Delta_f^-(A)$. But by (2) $\Delta_f^-(\Delta_f(A)) = \Delta_f(A)$, since $\Delta_f(A)$ is fully $f_{\setminus S}$-invariant, and $A \subset \Delta_f(A)$ and it follows that $\Delta_f^-(A) \subset \Delta_f^-(\Delta_f(A)) = \Delta_f(A)$. Therefore $\Delta_f(A) = \Delta_f^-(A)$.

(4) Assume $A \cap \Delta_f(B) \neq \varnothing$ and let $x \in A \cap \Delta_f(B)$. By (3) $x \in \Delta_f^-(B)$ and so $f^n(x) \in B$ for some $n \geq 0$, thus $f^n(x) \in X \cap f^n(A)$ and hence by Lemma 2 $f^n(x) \in A$, i.e., $f^n(x) \in A \cap B$. Therefore $A \cap B \neq \varnothing$ whenever $A \cap \Delta_f(B) \neq \varnothing$. But $\Delta_f(B)$ is $f_{\setminus S}$-invariant, and applying this to the sets A and $\Delta_f(B)$ then shows that $A \cap \Delta_f(B) \neq \varnothing$ whenever $\Delta_f(A) \cap \Delta_f(B) \neq \varnothing$. Finally, if $A \cap B \neq \varnothing$ then clearly both $A \cap \Delta_f(B) \neq \varnothing$ and $\Delta_f(A) \cap \Delta_f(B) \neq \varnothing$. $\square$

Lemma 7 *For each transitive cascade γ the domain D_γ is an element of $\mathcal{D}_f^\diamond$.*

Proof If $V \in \mathcal{I}_f^\diamond$ with $V \cap \Delta_f^\diamond(D_\gamma) \neq \varnothing$ then by Lemma 5 (7) $V \cap \Delta_f(D_\gamma) \neq \varnothing$ and hence by Lemma 6 (4) $V \cap D_\gamma \neq \varnothing$. It follows that $\Delta_f^\diamond(D_\gamma) \in \mathcal{S}_\gamma$, and thus that $D_\gamma = \Delta_f^\diamond(D_\gamma)$ (since $D_\gamma \subset \Delta_f^\diamond(D_\gamma)$). But $\Delta_f^\diamond(D_\gamma) \in \mathcal{D}_f^\diamond$ and so $D_\gamma \in \mathcal{D}_f^\diamond$. $\square$

Lemma 8 *If* $\gamma = \{U_n\}_{n\geq 1}$ *is a transitive cascade then* $\Delta_f^\diamond(U_n) = D_\gamma$ *for each* $n \geq 1$.

Proof Consider n to be fixed. By Lemma 7 we have $\Delta_f^\diamond(U_n) \subset \Delta_f^\diamond(D_\gamma) = D_\gamma$. Now put $U' = X \setminus \overline{\Delta_f(U_n)}$ (and so U' is either empty or an element of $\mathcal{I}_f^\diamond$) and suppose the regular open set $W = D_\gamma \setminus \overline{\Delta_f(U_n)} = D_\gamma \cap U'$ is non-empty. Then we have $D_\gamma \cap U' \in \mathcal{I}_f^\diamond$, thus $U_m \subset D_\gamma \cap U'$ for some $m \geq 1$ and in particular $U_m \subset U'$. However, $U_n \subset \overline{\Delta_f(U_n)} = X \setminus U'$, which is not possible, since either $U_n \subset U_m$ or $U_m \subset U_n$. Hence $W = \varnothing$, i.e., $D_\gamma \subset \overline{\Delta_f(U_n)}$, which implies that $D_\gamma = \mathrm{int}(D_\gamma) \subset \Delta_f^\diamond(U_n)$. □

Lemma 9 *If* $D \in \mathcal{D}_f^\diamond$ *is minimal then* $U_1 \cap U_2 \neq \varnothing$ *for all elements* U_1, U_2 *of* $\mathcal{I}_f$ *contained in* D.

Proof If $U \in \mathcal{I}_f$ with $U \subset D$ then $\Delta_f(U) \subset D$, thus $\Delta_f^\diamond(U) \subset D^\diamond = D$, which implies that $\Delta_f^\diamond(U) = D$, since D is minimal. Hence $\Delta_f^\diamond(U_1) = \Delta_f^\diamond(U_2)$, and therefore by Lemmas 5 (7) and 6 (4) it follows that $U_1 \cap U_2 \neq \varnothing$. □

We now start on the proof of Theorem 2. Arbitrary intersections of $f_{\setminus S}$-invariant sets are $f_{\setminus S}$-invariant and X is $f_{\setminus S}$-invariant and hence for each $A \subset X$ there is a least $f_{\setminus S}$-invariant set containing A, which will be denoted by $\Delta_f^+(A)$. If O is open then by Lemma 4 $\Delta_f^+(O)$ is also open, since then $\mathrm{int}(\Delta_f^+(O))$ is an $f_{\setminus S}$-invariant set containing O.

Suppose first $D \in \mathcal{D}_f^\diamond$ is minimal. Since the topology on X has a countable base then so does the subspace topology on D. There thus exists a sequence $\{O_m\}_{m\geq 1}$ of non-empty open subsets of D such that each non-empty open subset of D contains some O_m. For each $m \geq 1$ put $V_m = (\Delta_f^+(O_m))^\diamond$; then $V_m \in \mathcal{I}_f^\diamond$ and $V \subset D$. Now for each $n \geq 1$ put $U_n = \bigcap_{m=1}^n V_m$; then by Lemma 9 $U_n \neq \varnothing$ and so $U_n \in \mathcal{I}_f^\diamond$. Hence $\{U_n\}_{n\geq 1}$ is a decreasing sequence of sets from $\mathcal{I}_f^\diamond$ with $U_1 \subset D$. Consider $V \in \mathcal{I}_f^\diamond$ with $V \cap U_1 \neq \varnothing$; then $O_m \subset V \cap U_1$ for some $m \geq 1$ and so $V_m \subset V \cap U_1$, since $V \cap U_1 \in \mathcal{I}_f^\diamond$. In particular, $U_m \subset V$, since $U_m \subset V_m$, and this shows that $\gamma = \{U_n\}_{n\geq 1}$ is a transitive cascade. Moreover, $D \cap D_\gamma \supset U_1 \neq \varnothing$ and so $D \subset D_\gamma$. On the other hand, by Lemma 8 $D_\gamma = \Delta_f^\diamond(U_1)$, and $\Delta_f^\diamond(U_1) \subset D$, and hence $D_\gamma \subset D$. It follows that $D_\gamma = D$, which shows that each minimal element of $\mathcal{D}_f^\diamond$ is the domain of a transitive cascade.

Conversely, let $\gamma = \{U_n\}_{n\geq 1}$ be a transitive cascade and by Lemma 7 $D_\gamma \in \mathcal{D}_f^\diamond$. Consider $E \in \mathcal{D}_f^\diamond$ with $E \subset D_\gamma$. Then $U_m \subset E$ for some m, thus $\Delta_f^\diamond(U_m) \subset E$ and so by Lemma 8 $D_\gamma = \Delta_f^\diamond(U_m) \subset E$. This shows that D_γ is minimal.

This completes the proof of Theorem 2. □

If U is a transitive element of $\mathcal{I}_f^\diamond$ then Theorem 2 implies in particular that the domain D_U of U is a minimal element of $\mathcal{D}_f^\diamond$. Moreover, by Lemma 8 $D_U = \Delta_f^\diamond(U)$, and by Lemma 6 (3) $\Delta_f(U) = \Delta_f^-(U)$.

If γ and γ' are transitive cascades then by Theorem 2 the domains D_γ and $D_{\gamma'}$, being minimal elements of $\mathcal{D}_f^\diamond$, are either equal or disjoint. We next look at the relationship between γ and γ' when $D_\gamma = D_{\gamma'}$.

Lemma 10 *Let $\gamma = \{U_n\}_{n\geq 1}$ and $\gamma' = \{U'_n\}_{n\geq 1}$ be transitive cascades such that $D_\gamma = D_{\gamma'}$. Then the sequences γ and γ' are mutually cofinal, meaning that for each $p \geq 1$ there exist q, $r \geq 1$ with $U_q \subset U'_p$ and $U'_r \subset U_p$. In particular, we then have $C_\gamma = C_{\gamma'}$.*

Proof Let $p \geq 1$. By Lemma 8 $\Delta_f^\diamond(U_p) \cap D_{\gamma'} = D_\gamma \cap D_{\gamma'} \neq \varnothing$ and hence by Lemmas 5 (7) and 6 (4) $U_p \cap D_{\gamma'} \neq \varnothing$. Therefore $U'_r \subset U_p$ for some $r \geq 1$ and in the same way $U_q \subset U'_p$ for some $q \geq 1$, i.e., the sequences γ and γ' are mutually cofinal. $\square$

Transitive cascades γ and γ' will be called *equivalent* if the sequences γ and γ' are mutually cofinal. Lemma 10 implies that this is the case if and only if $D_\gamma = D_{\gamma'}$. By Theorem 2 each minimal element of $\mathcal{D}_f^\diamond$ contains a unique equivalence class of transitive cascades. For each transitive cascade $\gamma = \{U_n\}_{n\geq 1}$ put

$$\underline{\Delta}_f^-(\gamma) = \bigcap_{n\geq 1} \Delta_f^-(U_n) \ .$$

Therefore $x \in \underline{\Delta}_f^-(\gamma)$ if and only if for each $n \geq 1$ there exists an $m \geq 1$ such that $f^m(x) \in U_n$. By Lemma 6 (3) $\Delta_f^-(U_n) = \Delta_f(U_n)$ for each n, and hence $\underline{\Delta}_f^-(\gamma)$ is a fully $f_{\setminus S}$-invariant G_δ-set with $\underline{\Delta}_f^-(\gamma) \subset D_\gamma$. Moreover, by Lemma 8 $\Delta_f^\diamond(U_n) = D_\gamma$ for each $n \geq 1$, which implies that

$$D_\gamma \setminus \underline{\Delta}_f^-(\gamma) = \bigcup_{n\geq 1} (\Delta_f^\diamond(U_n) \setminus \Delta_f(U_n)) \subset \bigcup_{n\geq 1} \partial\Delta_f(U_n) \ ,$$

i.e., $D_\gamma \setminus \underline{\Delta}_f^-(\gamma)$ is contained in the meagre set $\bigcup_{n\geq 1} \partial\Delta_f(U_n)$. If U_m is transitive for some m (and so $U_n = U_m$ for all $n \geq m$) then $\underline{\Delta}_f^-(\gamma) = \Delta_f^-(U_m)$, and in this case $D_\gamma \setminus \underline{\Delta}_f^-(\gamma)$ is contained in the nowhere dense set $\partial\Delta_f(U_m)$. If γ and γ' are equivalent then clearly $\underline{\Delta}_f^-(\gamma) = \underline{\Delta}_f^-(\gamma')$; on the other hand, if γ and γ' are not equivalent then by Lemma 10 $\underline{\Delta}_f^-(\gamma) \cap \underline{\Delta}_f^-(\gamma') = \varnothing$.

We now give a characterisation of the transitive elements in $\mathcal{I}_f^\diamond$ which corresponds to a standard result concerning the property of being topologically transitive (defined in terms of closed rather than open sets), and which can by found in Walters [12]. For this we need an explicit expression for the least $f_{\setminus S}$-invariant set $\Delta_f^+(A)$ containing A.

Lemma 11 *For each $A \subset X$*

$$\Delta_f^+(A) = \{x \in X : x = f^n(y) \text{ for some } y \in A \text{ and some } n \geq 0\} \,.$$

Proof Denote the set $\{x \in X : x = f^n(y)$ for some $y \in A$ and some $n \geq 0\}$ by A' for each $A \subset X$. We first show that A' is $f_{\setminus S}$-invariant for each A. Let $x \in A' \setminus S$; then $f(x) \in X$ and $x \in f^n(A)$ for some $n \geq 0$, thus $f(x) \in X \cap f^{n+1}(A)$ and hence $f(x) \in A'$. Therefore $f_{\setminus S}(A') \subset A'$, i.e., A' is $f_{\setminus S}$-invariant. Next note that if B is $f_{\setminus S}$-invariant then by Lemma 2 (2) $X \cap f^n(B) \subset B$ for all $n \geq 1$ and therefore $B \subset B' = X \cap \bigcup_{n \geq 0} f^n(B) \subset B$; i.e., $B = B'$. Now let $A \subset X$ and B be $f_{\setminus S}$-invariant with $A \subset B$; then $A' \subset B' = B$, and therefore $A' = \Delta_f^+(A)$, since A' is $f_{\setminus S}$-invariant and contains A. □

In particular $\Delta_f^+(x) = \{y \in X : y = f^n(x)$ for some $n \geq 0\}$ for each $x \in X$. Thus if $x \in \Lambda_f(S) = X \setminus \Delta_f^-(S)$ then $\Delta_f^+(x)$ is the set of points in the orbit of x under f, and if $x \in \Delta_f^-(S)$ then $\Delta_f^+(x)$ is the finite set of points in the orbit of x under f, but omitting the element $\bullet$.

Proposition 3 *The following are equivalent for an element $U \in \mathcal{I}_f^\diamond$:*

(1) U is transitive.

(2) $U \subset (\Delta_f^-(O))^\diamond$ for each non-empty open subset O of U.

(3) $U = U_*^\diamond$, where $U_* = \{x \in U : U = (\Delta_f^+(x))^\diamond\}$.

(4) $U = (\Delta_f^+(x))^\diamond$ for some $x \in U$.

Proof (1) $\Rightarrow$ (2): By Lemmas 6 (1) and 4 the set $X \setminus \overline{\Delta_f^-(O)} = \operatorname{int}(X \setminus \Delta_f^-(O))$ is $f_{\setminus S}$-invariant and so $V = U \setminus \overline{\Delta_f^-(O)} = U \cap (X \setminus \overline{\Delta_f^-(O)}\,)$ is either empty or an element of $\mathcal{I}_f^\diamond$. But $O \cap V = O \cap (U \setminus \overline{\Delta_f^-(O)}) = O \setminus \overline{\Delta_f^-(O)} = \varnothing$ and it follows that $O \subset U \setminus \overline{V}$. In particular $U \neq V$, which implies $V = \varnothing$, i.e., $U \subset \overline{\Delta_f^-(O)}$. Thus $U \subset (\Delta_f^-(O))^\diamond$.

(2) $\Rightarrow$ (3): Since the topology on X has a countable base so does the subspace topology on U. There thus exists a sequence $\{O_m\}_{m \geq 1}$ with each O_m a non-empty open subset of U such that each non-empty open subset of U contains some O_m. Let $x \in U$; then $U = (\Delta_f^+(x))^\diamond$ if and only if $\Delta_f^+(x) \cap O_m \neq \varnothing$ for each $m \geq 1$, and by Lemma 11 this holds if and only if $x \in \Delta_f^-(O_m)$ for each $m \geq 1$. This shows that $U_* = U \cap \bigcap_{m \geq 1} \Delta_f^-(O_m)$ (and in particular U_* is a G_δ-set). But for each $m \geq 1$ we have $U \subset (\Delta_f^-(O_m))^\diamond$ and hence

$$U \cap \Delta_f^-(O_m) = U \setminus (U \cap ((\Delta_f^-(O_m))^\diamond \setminus \Delta_f^-(O_m)) \supset U \setminus \partial \Delta_f^-(O_m) \,.$$

It follows that $(X \setminus \overline{U}) \cup (U \cap \Delta_f^-(O_m)) \supset X \setminus (\partial U \cup \partial \Delta_f^-(O_m))$, which implies that $(X \setminus \overline{U}) \cup (U \cap \Delta_f^-(O_m))$ is a dense open subset of X. Therefore by the Baire category theorem the intersection

$$(X \setminus \overline{U}) \cup U_* = (X \setminus \overline{U}) \cup \Big(U \cap \bigcap_{m \geq 1} \Delta_f^-(O_m)\Big) = \bigcap_{m \geq 1} ((X \setminus \overline{U}) \cup (U \cap \Delta_f^-(O_m)))$$

is also dense and so

$$X = \overline{(X \setminus \overline{U}) \cup U_*} = \overline{(X \setminus \overline{U})} \cup \overline{U_*} = (X \setminus U^\diamond) \cup \overline{U_*} = (X \setminus U) \cup \overline{U_*} \ .$$

This shows that $U \subset \overline{U_*}$, i.e., $U \subset U_*^\diamond$.

(3) $\Rightarrow$ (4): This is clear, since $U \cap U_* \neq \varnothing$.

(4) $\Rightarrow$ (1): Let $V \in \mathcal{I}_f^\diamond$ with $V \subset U$ and let $x \in U$ be such that $U = (\Delta_f^+(x))^\diamond$. There thus exists $m \geq 0$ with $f^m(x) \in V$ and then $\Delta_f^+(f^m(x)) \subset V$, since V is $f_{\setminus S}$-invariant. Put $B = \{x, f(x), \ldots, f^{m-1}(x)\}$; then

$$U = (\Delta_f^+(x))^\diamond \subset \overline{B \cup \Delta_f^+(f^m(x))} = B \cup \overline{\Delta_f^+(f^m(x))} \subset B \cup \overline{V} \ ,$$

and so $U \setminus B' \subset \overline{V}$ with $B' = B \setminus \overline{V}$. It follows that $U \subset V$, since B' is finite and X is perfect. (If W is open with $W \cap U \neq \varnothing$ then $W \cap (U \setminus B') \neq \varnothing$ and so $W \cap V \neq \varnothing$, since $U \setminus B' \subset \overline{V}$. Hence $U \subset \operatorname{int}(\overline{V}) = V^\diamond = V$.) $\square$

We now turn to another decomposition of X into disjoint elements of $\mathcal{D}_f$ which more directly involves the set S. Each subset C of S is $f_{\setminus S}$-invariant, since $f_{\setminus S}(C) = f(C \setminus S) = \varnothing$, and therefore by Lemma 6 (3) $\Delta_f(C) = \Delta_f^-(C)$, i.e.,

$$\Delta_f(C) = \{x \in X : f^n(x) \in C \text{ for some } n \geq 0\} \ .$$

In particular, X is the disjoint union of the sets $\Delta_f(S)$ and $\Lambda_f(S)$, where as before $\Lambda_f(S) = \{x \in X : f^n(x) \notin S \text{ for all } n \geq 0\}$.

Put $\mathrm{Z}_f = \operatorname{int}(\Lambda_f(S))$ and $\Sigma_f = \Delta_f^\diamond(S)$. Then we have $\mathrm{Z}_f \in \mathcal{D}_f$, Σ_f is either empty or an element of $\mathcal{D}_f^\diamond$, the sets Z_f and Σ_f are disjoint and their union $\mathrm{Z}_f \cup \Sigma_f$ is a dense open subset of X, since $X \setminus (\mathrm{Z}_f \cup \Sigma_f) = \partial\, \overline{\Delta_f(S)}$ and the boundary of a closed set is nowhere dense. If D is a minimal element of $\mathcal{D}_f^\diamond$ then either $D \subset \Sigma_f$ or $D \cap \Sigma_f = \varnothing$ (in which case $D \subset Z_f^\diamond$).

Theorem 3 *Suppose S is finite and $\Sigma_f \neq \varnothing$ (and so S is non-empty). Then Σ_f contains at least one and at most $\#(S)$ minimal elements of $\mathcal{D}_f^\diamond$ (with $\#(S)$ the cardinality of S).*

Proof This requires some preparation.

Lemma 12 *If $C \subset S$ and $U \in \mathcal{I}_f$ with $U \cap \Delta_f^\diamond(C) \neq \varnothing$ then $U \cap C \neq \varnothing$. In particular, if $\Delta_f^\diamond(C) \neq \varnothing$ then $\Delta_f^\diamond(C)$ contains an element of C, and each $U \in \mathcal{I}_f^\diamond$ with $U \subset \Sigma_f$ contains an element of S.*

Proof Put $V = U \cap \Delta_f^\diamond(C)$. Then by Lemma 5 (7) $V \cap \Delta_f(C) \neq \varnothing$. Thus, since $\Delta_f(C) = \Delta_f^-(C)$, there exists $c \in C$, $x \in V$ and $n \geq 0$ with $f^n(x) = c$ and it follows that $c = f^n(x) \in X \cap f^n(V)$. But by Lemma 3 $X \cap f^n(V)$ is open and $X \cap f^n(V) \subset X \cap f^n(U) \subset U$. Hence $c \in U$, which shows $U \cap C \neq \varnothing$. □

Proposition 4 *If $c \in S$ with $\Delta_f^\diamond(c) \neq \varnothing$ then $\Delta_f^\diamond(c)$ is a minimal element of $\mathcal{D}_f^\diamond$ containing c.*

Proof By Lemma 12 $c \in \Delta_f^\diamond(c)$. Let $D \in \mathcal{D}_f^\diamond$ with $D \subset \Delta_f^\diamond(c)$. Then, again by Lemma 12 $c \in D$, hence $\Delta_f(c) \subset D$, since $\Delta_f(c)$ is the least fully $f_{\setminus S}$-invariant set containing c, and then $\Delta_f^\diamond(c) \subset D^\diamond = D$. Thus $D = \Delta_f^\diamond(c)$, which shows that $\Delta_f^\diamond(c)$ is minimal. □

Lemma 13 *Let $C_1, \ldots, C_n$ be subsets of S and put $C = \bigcup_{k=1}^n C_k$. Then*

$$\bigcup_{k=1}^n \Delta_f^\diamond(C_k) \subset \Delta_f^\diamond(C) \subset \bigcup_{k=1}^n \Delta_f^\diamond(C_k) \cup N$$

with N the nowhere dense set $\bigcup_{k=1}^n \partial\,\overline{\Delta_f(C_k)}$.

Proof Since $C_k \subset C$ it immediately follows that $\Delta_f^\diamond(C_k) \subset \Delta_f^\diamond(C)$ and hence that $\bigcup_{k=1}^n \Delta_f^\diamond(C_k) \subset \Delta_f^\diamond(C)$. On the other hand $\Delta_f(C) = \bigcup_{k=1}^n \Delta_f(C_k)$ and thus

$$\begin{aligned}\Delta_f^\diamond(C) \subset \overline{\Delta_f(C)} &= \bigcup_{k=1}^n \overline{\Delta_f(C_k)} = \bigcup_{k=1}^n \big(\Delta_f^\diamond(C_k) \cup \big(\overline{\Delta_f(C_k)} \setminus \Delta_f^\diamond(C_k)\big)\big) \\ &= \bigcup_{k=1}^n \big(\Delta_f^\diamond(C_k) \cup \partial\,\overline{\Delta_f(C_k)}\big) = \bigcup_{k=1}^n \Delta_f^\diamond(C_k) \cup N\,. \quad \square\end{aligned}$$

We now come to the proof of Theorem 3. First, since $S = \bigcup_{c \in S}$ it follows from Lemma 13 that $\bigcup_{c \in S} \Delta_f^\diamond(s) \subset \Sigma_f \subset \bigcup_{c \in S} \Delta_f^\diamond(c) \cup N$, where N is nowhere dense. Thus, since Σ_f is non-empty and open, there exists a $c \in S$ such that $\Delta_f^\diamond(c) \neq \varnothing$, and then by Proposition 4 $\Delta_f^\diamond(c)$ is a minimal element of $\mathcal{D}_f^\diamond$, which shows that Σ_f contains at least one minimal element of $\mathcal{D}_f^\diamond$. Now let D be any minimal element of $\mathcal{D}_f^\diamond$ contained in Σ_f. Then $D \cap \Delta_f^\diamond(c) \neq \varnothing$ for some $c \in S$ (since N is nowhere dense) and then $D = \Delta_f^\diamond(c)$, since by Proposition 4 $\Delta_f^\diamond(c)$ is also a minimal element of $\mathcal{D}_f^\diamond$. Therefore Σ_f contains at most $\#(S)$ minimal elements of $\mathcal{D}_f^\diamond$. This completes the proof of Theorem 3. □

Theorems 2 and 3 together show that if S is finite and $\Sigma_f \neq \varnothing$ then Σ_f contains at least one and at most $\#(S)$ equivalence classes of minimal cascades.

Proposition 5 *If $D \in \mathcal{D}_f^\diamond$ with $D \subset \Sigma_f$ then $D = \Delta_f^\diamond(S \cap D)$ for each $D \in \mathcal{D}_f^\diamond$.*

Proof First consider an element $U \in \mathcal{I}_f^\diamond$ with $U \subset \Sigma_f$. Then, since $S \cap U \subset U$, it follows that $\Delta_f(S \cap U) \subset \Delta_f(U)$ and therefore that $\Delta_f^\diamond(S \cap U) \subset \Delta_f^\diamond(U)$. Now if the $f_{\setminus S}$-invariant open set $V = U \setminus \overline{\Delta_f(S \cap U)}$ were non-empty then by Lemma 12 it would contain an element of $S \cap U$. Since $S \cap U \subset \overline{\Delta_f(S \cap U)}$ this is not possible, and so $V = \varnothing$, i.e., $U \subset \overline{\Delta_f(S \cap U)}$. Hence $\Delta_f(U) \subset \overline{\Delta_f(S \cap U)}$, since $\overline{\Delta_f(S \cap U)}$ is fully $f_{\setminus S}$-invariant, and thus also $\Delta_f(U) \subset \Delta_f^\diamond(S \cap U)$. This shows $\Delta_f^\diamond(U) = \Delta_f^\diamond(S \cap U)$. In particular, if $D \in \mathcal{D}_f^\diamond$ with $D \subset \Sigma_f$ then $D = \Delta_f^\diamond(S \cap D)$, since here $D = \Delta_f^\diamond(D)$. $\square$

By Proposition 5 the mapping $D \mapsto S \cap D$ from the set $\{D \in \mathcal{D}_f^\diamond : D \subset \Sigma_f\}$ to the set of subsets of S is injective.

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